Pathways to Research
EDUCATION

Exploring Elementary Mathematical Writing

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Educators have long recommended writing as a way to engage elementary students in mathematical communication (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 1991). The NCTM (2000) identified communication as one of the five process standards and an essential component of mathematics instruction. Further extending the emphasis on communication, the Common Core State Standards for Mathematics (CCSS-M; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) recommended that students learn to communicate precisely to others, construct mathematical arguments, and read and critique the mathematical reasoning of others. The emphasis on mathematical communication in the standards documents highlights the need to engage elementary students in mathematical writing.

Mathematical writing is distinct from other writing genres because it requires the combination of mathematics content knowledge, general writing skills, and skills unique to the mathematics discipline, such as the use of mathematical representations, precise mathematical language, and the ability to combine equations and expressions with words (Namkung et al., 2020). Mathematical writing reflects the styles of communication used by the mathematics community, including the use of diagrams and symbols (see Figure 1). Researchers have found that writing to learn mathematics increases students' learning of mathematics and improves students' ability to communicate mathematically (Graham et al., 2020; Bangert-Drowns et al., 2004). Further, students who wrote in mathematics class demonstrated improved mathematics achievement on posttests over pretest scores (Cohen et al., 2015; Tan & Garces-Bascal, 2013; Kostos & Shin, 2010; Cross, 2009).

Mathematical writing also affords students unique benefits for learning. Writing provides students an opportunity to make their mathematical ideas permanent—allowing students to "see" their thinking (Colonnese et al., 2018) and engage in metacognitive thinking (Pugalee, 2004). Writing can engage students in the active construction of understanding (Cross, 2009) and increase students' use of mathematically precise terms (Cohen et al., 2015). By sharing their writing, students can identify new ways of thinking about mathematics and have an opportunity to state their ideas clearly and convincingly (Liedtke & Sales, 2001).

Mathematical writing has become more prevalent in the US, yet it still remains underutilized as a way to communicate mathematically (Powell et al., 2021; Kosko,

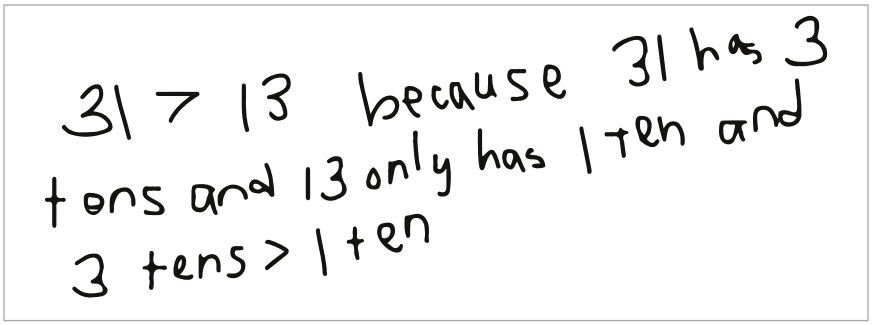

Figure 1: This example of student writing represents the combination of general writing skills and those specific to the mathematics discipline.

Notice how the student has used the greater than symbol to compare 31 and 13 and the 3 tens and 1 ten.

2016). Teachers have shared that curriculum demands and/or the time needed for writing contribute to the low frequency of mathematical writing (Fukawa-Connelly & Buck, 2010). As a result, elementary students may have limited opportunities to learn how to write mathematically and to use writing to communicate their mathematical thinking. Consequently, students may not attain the maximum benefits from engaging in writing.

In addition to the underutilization of mathematical writing, many different kinds of writing have been recommended in practitioner articles, books about writing in mathematics, and in research articles (Colonnese, 2020). Bosse and Faulconer (2008) attribute the ambiguity around mathematical writing to the failure "to distinguish reading and writing about mathematics from reading and writing in mathematics" (p. 10). Reading and writing about mathematics forefronts students' literacy, whereas reading and writing in mathematics are intended to advance students' mathematical knowledge. Some examples of writing taking place in math class have included writing a narrative story with a reference to a mathematics problem (Carter, 2009), describing who's who in mathematics (Rothstein et al., 2003), writing a mathematical argument (Kiuhara et al., 2020), and writing proofs (Güler & Dikici, 2014).

In analyzing these examples more closely, writing about a mathematician (who's who in mathematics) and writing a narrative story are specific to writing in the language arts—biographies and stories are not typical products of mathematical analysis. These examples also do not generally elicit students' use of mathematical reasoning or engage them in communicating mathematically. On the other hand, writing a mathematical argument and proof-writing can encourage mathematical reasoning and justification and provide students an opportunity to utilize the language of mathematics (Whitenack & Yackel, 2002).

The lack of clarity about how elementary students should engage in writing during mathematics class leads to the question of what kinds of mathematical writing can advance students' mathematics learning and ability to communicate mathematically. We will address this question by considering the kinds of writing that position students to think, reason, and communicate mathematically. Further clarifying the kinds of writing that support student learning can help educators make informed choices about the ways they are engaging students in writing during mathematics class.

Defining Mathematical Writing

The Elementary Mathematical Writing Task Force attended to the lack of clarity related to writing in mathematics class by identifying types and purposes that would support students' learning of mathematics (Casa et al., 2016). The task force identified two overarching goals for mathematical writing: "for students to reason mathematically and to communicate ideas" (Casa et al., 2016, p. 3). The goals for mathematical writing reflect essential components of learning and doing mathematics. Reasoning is a defining characteristic of mathematics learning (NCTM, 2000), and mathematical communication has long been documented as a way for students to develop a deeper understanding of mathematics (e.g., Hufferd-Ackles et al., 2004; Carpenter et al., 2003).

Further expanding on the two overarching goals, the task force recommended four types of mathematical writing and the purposes each type of writing serves. The four types of writing include exploratory, explanatory, argumentative, and mathematically creative writing (Casa et al., 2016). The types of mathematical writing students engage in are intended to facilitate the advancement of students' learning of mathematics. In the next sections, I describe the four types of mathematical writing in more detail and share related research.

Exploratory Writing

The purpose of exploratory writing is "to personally make sense of a problem, situation, or one's own ideas" (Casa et al., 2016, p. 4). Exploratory writing builds upon the ideas of exploratory talk (Barnes, 2008; Cazden, 2001) and rough draft thinking (Jansen, 2020). When students engage in exploratory talk, they are manipulating what they already know and using the resources available to them to make sense of their thoughts and work on understanding (Barnes, 2008). Similarly, when engaging in exploratory writing, students are writing to document partial, unfinished, or initial mathematical ideas to make sense of a mathematical concept (Firmender et al., 2017a). An important difference between exploratory writing and other kinds of writing is that the audience is the individual student. Students can then write in ways that make sense to them, including using their home language and drawing representations. Figure 2 is a representative example of an elementary student's exploratory writing.

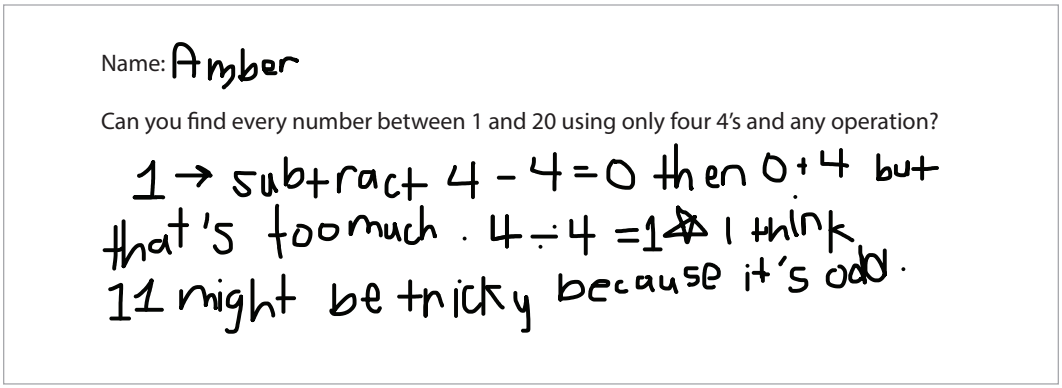

Figure 2: Amber engages in exploratory writing to help begin working on the Four 4's task.
Notice how Amber includes a partial idea that becomes a complete idea and a question that she is pondering as she works through the task.

Explanatory Writing

Explanatory writing, serving to have students write to describe or explain (Casa et al., 2016), is one of the more common types of mathematical writing teachers incorporate into daily mathematics instruction and is included in mathematics assessments (Kosko, 2016; Powell et al., 2017, 2021). In a study of the grade three student books in the nine most frequently used mathematics programs (i.e., *Investigations*, *Everyday Mathematics*, *GoMath*), Casa et al. (2019) found that the majority of prompts (65 percent) that elicited writing asked students to provide explanations. An example of one prompt asking students to explain is, "Amy wanted to give three of her friends some candy. If she gives each friend four candies, how many candies will she need? Explain how you got your answer." Students may interpret this prompt as asking for a restatement of their procedure in words. While writing about a procedure may be helpful for students to further understand their problem-solving process, these types of prompts can limit the potential for students to engage in writing to reason mathematically.

To engage students in writing an explanation that evokes mathematical reasoning, prompts should be developed to attend to higher levels of thinking (Firmender et al., 2017a). A higher-level cognitive demand task includes ones that encourage students to use procedures for developing a deeper understanding of mathematics, draw connections, engage with conceptual ideas, notice patterns, and explain generalizations (Smith & Stein, 1998). Figure 3 is a representation of an elementary student's explanatory writing. The task is encouraging students to draw connections between the two's facts and the four's facts. This is a helpful strategy for developing fluency with multiplication facts. The task also positions the student to write to a friend of their choosing. The outside audience can help encourage students to provide more detail and use mathematically precise language in their response because they are explaining the concept to someone who is not in their class (Colonnese et al., 2018).

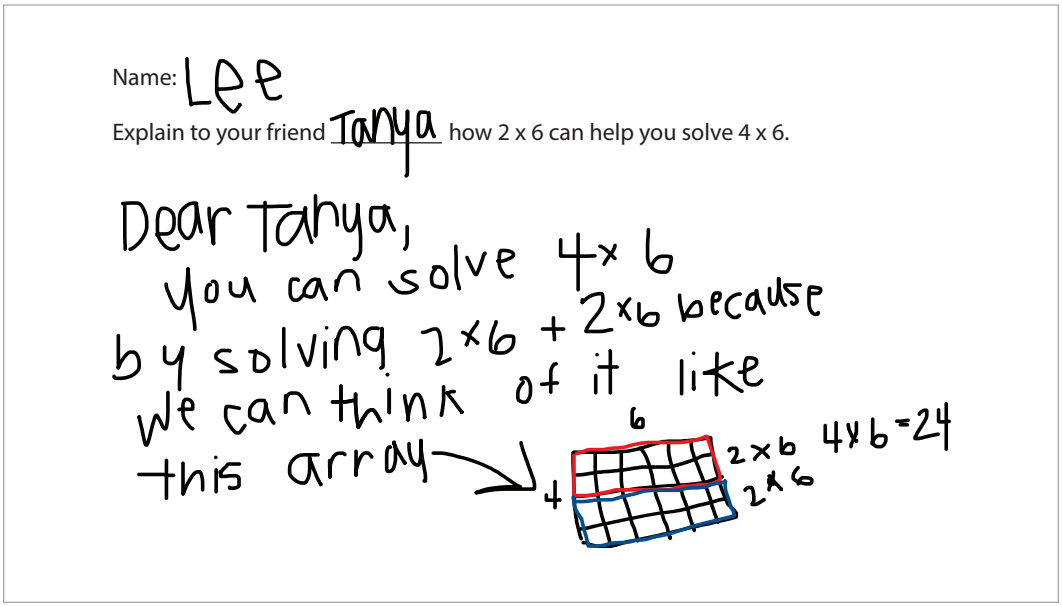

Figure 3: **The student, Lee, writes an explanation to describe how their friend can use 2 x 6 to solve 4 x6.** Lee included a representation to support their explanation.

Argumentative Writing

The purpose for argumentative writing reflects the third Standard for Mathematical Practice (SMP 3) in the CCSS-M (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010): "Construct viable arguments and critique the reasoning of others" (p. 6), and builds on the NCTM's (2000) emphasis on reasoning, proof, and communication. Much of the recent research in mathematics education related to argumentation has been informed by Toulmin's (1958/2003) model of an argument (Connor et al., 2014; Inglis et al., 2007). Toulmin (1958/2003) described the purpose of an argument as proving a claim true to an audience. Pallanck et al. (2020) adapted Toulmin's model for elementary students to include a claim (the position taken), evidence (support for the claim), and warrant (connecting the evidence and claim). Figure 4 is an example of a student's mathematical argument. Miles writes a critique of Dan's argument. Miles's claim is that he disagrees with Dan. The evidence is that $6 - 4 = 2$ and $4 - 6 = -2$. The warrant connecting the claim and the evidence is that 2 does not equal –2.

Researchers have identified argumentation as a major thinking skill, important for critical thinking and learning (Wagner et al., 2014). Earlier experiences with argumentative writing can support students in developing the skills needed to communicate their mathematical reasoning and build a foundation for writing mathematical proofs (Cervantes-Barraza et al., 2020; Connor et al., 2014). By the end of secondary school, students should be able to understand and write mathematical

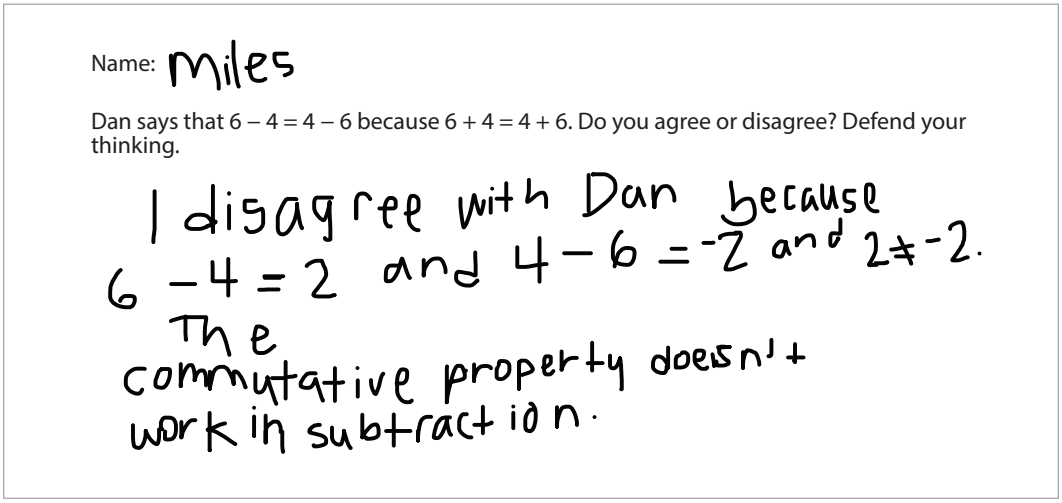

Figure 4: **Miles writes a critique of Dan's argument using a claim, evidence, and warrant.**

proofs (NCTM, 2000). However, students often find explaining their reasons in a mathematical argument challenging (Pallanck et al., 2020; Monoyiou et al., 2006; Evens & Houssart, 2004; Hoyles & Kuchemann, 2002). As a result, students need more instruction that attends to the components of a mathematical argument and opportunities to write mathematical arguments. Studies focused on instruction related to argumentation have included debate journals (Bostiga et al., 2016), the use of modeling (Pallanck et al., 2020), and a writer's workshop model (Kiuhara et al., 2020). In all three studies, the authors found that students' mathematical reasoning improved.

Mathematically Creative Writing

Mathematically creative writing "serves to encourage elementary students to think creatively and document their mathematical ideas that extend beyond the expected or intended outcome of a task, situation, or problem" (Casa et al., 2016, p. 7). Originality (ideas that are novel to the group of students), flexibility (adapting strategies to new contexts), and fluency (solving in multiple ways) are characteristics of mathematically creative writing (Firmender et al., 2017b). The environment the teacher creates and opportunities for engaging in mathematically creative writing are essential to encouraging this kind of writing (Firmender, 2021). The following example (Figure 5) is intended to help illustrate mathematically creative writing:

The students in Ms. W's class were learning to subtract two-digit numbers. Several students were using a variety of methods such as counting back, adding up, the number line, and base ten blocks. One student, Marco, came up to Ms. W and showed her his paper:

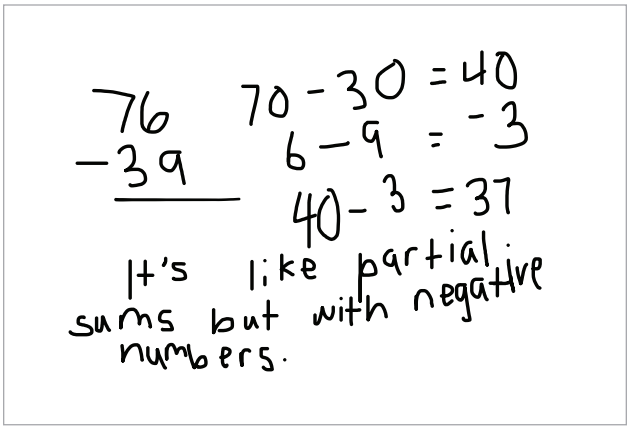

Figure 5: **Marco engages in mathematically creative writing by coming up with a novel approach for solving 76 – 39.**

The strategy Marco shares is mathematically creative because it is novel to his class, demonstrating originality and flexibility because he is adapting the partial sums strategy to solve an equation involving subtraction. Ms. W provided Marco an opportunity to engage in mathematically creative writing by allowing students to explore different strategies for subtraction and not requiring students to use a particular strategy. Writing allowed Marco to use expressions and words to help communicate his thinking and illustrate the connections he noticed to the partial sum strategy.

Mathematically creative writing is different from other types of creative writing, such as writing a story about a triangle's adventure, because the emphasis is on creativity within the mathematical ideas. As in Marco's writing, he is focusing on the mathematical concept of subtraction and discovering a novel strategy. It is important to note that, like the other types of writing, students need opportunities and encouragement to engage in mathematically creative writing.

Conclusion

The kinds of writing described in this article represent a vision for mathematical writing that engages students in writing for the purpose of communicating and reasoning mathematically. Students should write mathematically throughout elementary school, beginning in kindergarten (Kosko & Zimmerman, 2019; Casa et al., 2017). Resources and routines such as the instructional guidelines (Colonnese, 2020) or writer's workshops in mathematics (Martin, 2015) can support teachers with the implementation of mathematical writing. However, this remains an area in need of continued development. Mathematical writing can help to advance elementary students' learning of mathematics and ability to communicate mathematically, making it an important mode of communication to continue investigating.

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